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A SEARCH ON INTEGER SOLUTIONS TO QUINTIC EQUATION WITH TWO UNKNOWNS $x^2 - 2xy^2 = ky^5$

Dr. J.Shanthi[1*], Dr. M.A.Gopalan[2]

[1] Assistant Professor, Department of Mathematics,
Shrimati Indira Gandhi College,Affiliated to Bharathidasan University, Trichy-620 002
Tamil Nadu,India

[2] Professor, Department of Mathematics, Shrimati Indira Gandhi College,Affiliated to
Bharathidasan University, Trichy-620 002,
Tamil Nadu, India.

Abstract

The process of obtaining many integer solutions to non-homogeneous polynomial equation of degree five with two unknowns given by $x^2 - 2xy^2 = ky^5$ is illustrated. A few relations between the solutions are presented.

Keywords: Binary quintic equation, Non-homogeneous quintic equation ,Integer solutions

Notations

$$t_{m,n} = n[1 + \frac{(n-1)(m-2)}{2}]$$

$$P_n^5 = \frac{n^2(n+1)}{2}$$

I.Introduction

The theory of Diophantine equations is an ancient subject that typically involves solving, polynomial equation in two or more variables or a system of polynomial equations with the number of unknowns greater than the number of equations, in integers and occupies a pivotal role in the region of mathematics.The subject of Diophantine equations has fascinated and inspired both amateurs and mathematicians alike and so they merit special recognition. Solving higher degree

Diophantine equations can be challenging as they involve finding integer solutions that satisfy the given polynomial equation. Learning about the various techniques to solve these higher power Diophantine equation in successfully deriving their solutions help us understand how numbers work and their significance in different areas of mathematics and science. For the sake of clear understanding by the readers, one may refer the varieties of Quintic Diophantine equations with multi variables [1-11]. It seems that much work has not been done regarding polynomial Diophantine equations of degree five with two unknowns. This paper aims at determining many integer solutions to non-homogeneous polynomial equation of degree five with two unknowns given by $x^2 - 2xy^2 = ky^5$. A few relations between the solutions are presented. A procedure for obtaining second order Ramanujan numbers through integer solutions of the given binary quintic equation is illustrated.

II.Method of analysis

The non-homogeneous quinticequation with two unknowns under consideration is

$$x^2 - 2xy^2 = ky^5 \quad (1)$$

Treating (1) as a quadratic in x and solving for the same ,we have

$$x = y^2[1 \pm \sqrt{ky+1}] \quad (2)$$

Let

$$\alpha^2 = ky + 1 \quad (3)$$

which, after some algebra ,is satisfied by

$$y_0 = k\beta^2 + 2\beta, \alpha_0 = k\beta + 1 \quad (4)$$

Assume the second solution to (3) as

$$\alpha_1 = h - \alpha_0, y_1 = h + y_0 \quad (5)$$

where h is an unknown to be determined. Substituting (5) in (3) and simplifying, we have

$$h = 2\alpha_0 + k$$

and in view of (5) ,it is seen that

$$\alpha_1 = \alpha_0 + k , y_1 = y_0 + 2\alpha_0 + k$$

The repetition of the above process leads to the general solution to (3) as

$$\begin{aligned} \alpha_n &= \alpha_0 + k n = k\beta + 1 + k\, n \\ y_n &= y_0 + 2n\alpha_0 + kn^2 = k\beta^2 + 2\beta + 2n(k\beta + 1) + kn^2 \end{aligned} \tag{6}$$

From (2) ,we get

$$\begin{aligned} x_n &= y_n^2 [1 \pm \alpha_n] \\ &= (kn + 2 + k\beta)\, y_n^2 \ , -(kn + k\beta)\, y_n^2 \end{aligned}$$

Thus , we have two sets of integer solutions to (1) represented by

Set 1

$$\begin{aligned} x_n &= x_n(k,\beta) = (\beta + n)^2 (k\beta + kn + 2)^3 \ , \\ y_n &= y_n(k,\beta) = (\beta + n)\ (k\beta + kn + 2) \end{aligned}$$

Set 2

$$\begin{aligned} x_n &= x_n(k,\beta) = -k\ (\beta + n)^3 (k\beta + kn + 2)^2 \ , \\ y_n &= y_n(k,\beta) = (\beta + n)\ (k\beta + kn + 2) \end{aligned}$$

Considering set 1 ,the following relations are observed :

(i) $y_{n+2}(k,\beta) - 2y_{n+1}(k,\beta) + y_n(k,\beta) = 2\ k , n = 0,1,2,...$

(ii) $2x_n(k,\beta) + k\, y_n^3(k,\beta)$ is a perfect square

(iii) $k\, y_n(k,\beta) + 1$ is a perfect square

(iv) $k\, y_n^3(k,\beta) - 2x_n(k,\beta)$ is written as difference of two squares

(v) $\dfrac{x_n(k,\beta)}{y_n^2(k,\beta)} - 1$ is a perfect square when $\beta = \alpha^2 n^2 k + (2\alpha - 1)n$

(vi) $y_n^2(k,\beta)\,[\,y_n(k,\beta+2) - y_n(k,\beta) - 4k + 4\,] = 4\ x_n(k,\beta)$

(vii) $y_n^2(k,\beta)\,[\,y_n(k,\beta+2)-y_n(k,\beta+1)-3k+2\,]=2\;x_n(k,\beta)$

(viii) $y_n(k,\beta+2)-2\,y_n(k,\beta+1)+y_n(k,\beta)\equiv 2\,k$

(ix) $(k\beta+kn+2)\,[\,y_n(k,\beta+1)-k-2]=(k\beta+kn+2k+2)\;y_n(k,\beta)$

(x) $y_n(k,\beta)+(k-2)(\beta+n)=2k\,t_{3,\beta+n}$

(xi) $x_n(k,\beta)+y_n^2(k,\beta)=2\,(\beta+n)^2\,P^5_{k\beta+kn+2}$

(xii) $(\beta+n)\,x_n\,(k,\beta)=y_n^3(k,\beta)$

(xiii) From the integer solutions to (1) given by Set 1,one may generate Second order Ramanujan numbers.

Illustration :

$$y_2(2,1)=24=1*24=2*12=3*8=4*6$$
$$=\quad F_1 \qquad F_2 \qquad F_3 \qquad F_4$$

$$F_1=F_2\Rightarrow(24+1)^2+(12-2)^2=(24-1)^2+(12+2)^2$$
$$=25^2+10^2=23^2+14^2=725$$

$$F_1=F_3\Rightarrow(24+1)^2+(8-3)^2=(24-1)^2+(8+3)^2$$
$$=25^2+5^2=23^2+11^2=650$$

$$F_1=F_4\Rightarrow(24+1)^2+(6-4)^2=(24-1)^2+(6+4)^2$$
$$=25^2+2^2=23^2+10^2=629$$

$$F_2=F_3\Rightarrow(12+2)^2+(8-3)^2=(12-2)^2+(8+3)^2$$
$$=14^2+5^2=10^2+11^2=221$$

$$F_3=F_4\Rightarrow(8+3)^2+(6-4)^2=(8-3)^2+(6+4)^2$$
$$=11^2+2^2=5^2+10^2=125$$

Thus, 725,650 ,629 ,221 ,125 represent second order Ramanujan numbers .

A similar observation may be performed by considering the solutions Set 2.

III.Conclusion

The polynomial equation of degree five with two unknowns given by $x^2-2xy^2=ky^5$ has been studied to obtain non-zero integer solutions .The process of eliminating the square-root will be beneficial for the researchers. As quintic equations are plenty , one may attempt to determine the solutions in integers for other choices of quintic Diophantine equations.

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